

Matrix Element - Simple Illustration

From now on

- (i) ALWAYS in Interaction Picture
drop I subscript
- (ii) DROP HATS on operators

Simple Example

Tong §3.2
p53+

- Consider a theory with
- (i) ONE spin zero particle (ϕ) = own anti-particle
mass m with $\omega_k = \sqrt{k^2 + m^2}$
 - (ii) ONE particle (ψ) & One anti-particle $(\bar{\psi})$ (distinct)
charge $+1$ & -1 , mass M , spin 0
 $\omega_k = \sqrt{k^2 + M^2}$

Use

- (i) Real scalar field $\phi(x) = \phi^\dagger(x)$
- (ii) Complex scalar field $\psi(x) \neq \psi^\dagger(x)$

Usual free Hamiltonian

$$\mathcal{L}_0 = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 + |\partial_\mu \psi|^2 - M^2 |\psi|^2$$

EFS check $\psi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2)$
gives usual form

Interactions

$$H_{int} = g \int d^3x \underbrace{\phi(x) \psi^\dagger(x) \psi(x)}$$

c.c. = COUPLING
CONSTANT

- strength of interaction
- real

- Conserves $\# \psi - \# \bar{\psi}$
- $\# \phi$ not conserved

Theory called

SCALAR YUKAWA THEORY

= SYTh

(Yukawa theory describes n/p interaction via 'meson' ϕ)

Can we describe the decay of one ϕ into a $\psi/\bar{\psi}$ pair?

Need $\mathcal{M} = \langle f, t_f = +\infty | S | i, t_i = -\infty \rangle$

where

S Matrix

$$S := U(t_f = +\infty, t_i = -\infty)$$

(Scattering)

$$\approx 1 - ig \int_{-\infty}^{+\infty} d^3x \phi(x) \psi^\dagger(x) \psi(x) + O(g^2)$$

Free = NO scattering Term

STATES

- Our ~~experiments~~ ^{measurement} are conducted at $t_f = +\infty$
 $t_i = -\infty$
 since interactions happen on very short time scale

- At t_i, t_f NOTHING is happening
 particles widely separated

$\Rightarrow$ behave like free fields/particles

$\Rightarrow$ Use FREE field eigenstates to represent

$$|i, t_i = -\infty\rangle$$

$$\& |f, t_f = +\infty\rangle$$

STATES (cont)

Taney §2.4
p29+

We use the free fields to define a useful basis in our Hilbert space of states
ie $H_{int} = 0$ at $t = \pm \infty$.

(i) Vacuum $|0\rangle$ vacuum $\hat{a}_p |0\rangle = 0$ etc.

(ii) One-Particle

State: $|p\rangle = \hat{a}_p^\dagger |0\rangle$

(ii) Multi-particle

States $|p, q\rangle = \hat{a}_p^\dagger \hat{a}_q^\dagger |0\rangle$ 2 particle
etc.

Normalisation

(i) Vacuum $\langle 0|0\rangle = 1$ by definition

(ii) 1 particle states $\langle p|q\rangle = \langle 0|\hat{a}_p \hat{a}_q^\dagger |0\rangle$

$$= \langle 0|(\delta(p-q) + \hat{a}_p^\dagger \hat{a}_p) |0\rangle$$

$$= \delta^3(p-q)$$

etc.

Relativistic Normalisation

The normalisation above is NOT Lorentz invariant ^(L.inv.)
 which makes ^{relativistic} calculations complicated

We know that

$$1 = \int d^3p |p\rangle \langle p|$$

e.g. $1|q\rangle = \int d^3p |p\rangle \langle p|q\rangle \delta^3(p-q)$

LHS L.inv $\Rightarrow$ RHS L.inv.

EFS. But $\frac{\int d^3p}{2\omega_p} = \int d^3p \delta(p^2 - m^2) \Theta(p_0)$ IS LORENTZ INV
 NOT $\int d^3p$
 Tony p261
 e.o.L 16 24/11/14

(i) $\Rightarrow 2\omega_p \delta^3(p-q)$ NOT $\delta^3(p-q)$ is L.inv.

as $1 = \frac{\int d^3p}{2\omega_p} \cdot 2\omega_p \delta^3(p-q)$

(ii) $1 = \frac{\int d^3p}{2\omega_p} (\sqrt{2\omega_p} |p\rangle) \cdot (\sqrt{2\omega_p} \langle p|)$

$|p\rangle \cdot \langle p|$

$(\delta^3 > 1 \text{ factor})$

Define new states $|p\rangle = \sqrt{2\omega_p} |p\rangle = |\phi(p)\rangle$
 *** NOTATION: No $\sim$ under p for relativistic normalised states
 with normalisation $\langle p|q\rangle = 2\omega_p \delta^3(p-q)$

$1 = \frac{\int d^3p}{2\omega_p} |p\rangle \langle p|$
 [W.B. $\hat{A}_p = \sqrt{2\omega_p} \hat{a}_p$ is rel. charge too!]

Decay again

$$M \approx -ig \langle f | \int d^4x \psi^\dagger(x) \psi(x) \phi(x) | i \rangle + O(g^2) \\ = -ig \int d^4x \langle \text{left} | \text{right} \rangle$$

where

$$| \text{right} \rangle = \phi(x) | i \rangle$$

$$= \int \frac{d^3k}{\sqrt{2\omega_k}} \left(a_{\underline{k}} e^{-ikx} + a_{\underline{k}}^+ e^{+ikx} \right) \sqrt{2\omega_p} \hat{a}_p^+ | 0 \rangle$$

($k_0 = \omega_k$)

$a^+ a^+ | 0 \rangle =$ Two ϕ term. while $\langle f | \psi^\dagger \psi =$ terms contain zero ϕ

$\Rightarrow$ Overlap zero

$$\sim \langle 0 | \underbrace{(b's \& c's)}_{\text{commute}} a^+ a^+ | 0 \rangle = 0$$

$$\Rightarrow | \text{right} \rangle = \int \frac{d^3k}{\sqrt{2\omega_k}} \sqrt{\omega_p} e^{-ikx} \underbrace{\hat{a}_k \hat{a}_p^+}_{\left(a_p^+ a_k + \delta^3(k-p) \right)} | 0 \rangle$$

$\neq a^+ a | 0 \rangle = 0$

$$\langle \text{left} | = \langle f | \psi^\dagger(x) \psi(x)$$

$$= \langle 0 | b_{q_1} c_{q_2} \sqrt{2\omega_{q_1} 2\omega_{q_2}} \cdot \int \frac{d^3k}{\sqrt{2\omega_{k_1}}} \left(b_{k_1}^+ e^{+ik_1x} + c_{k_1} e^{-ik_1x} \right)$$

$\omega_k = k^2 + M^2$
 $\nearrow k_1^{u=0} = \pm \omega_{k_1}$ etc

$$\cdot \int \frac{d^3k_2}{\sqrt{2\omega_{k_2}}} \left(b_{k_2} e^{-ik_2x} + c_{k_2}^+ e^{+ik_2x} \right)$$

Again only $bb^+ cc^+$ term has zero $bdc \neq$ to match $| \text{left} \rangle$'s: $\psi \Delta \psi^\dagger \neq$

$$\text{EPS} \Rightarrow \langle \text{left} | = \langle 0 | \int d^3 \vec{k}_1 \int d^3 \vec{k}_2 \frac{\sqrt{\Omega_{q_1} \Omega_{q_2}}}{\sqrt{\Omega_{k_1} \Omega_{k_2}}} e^{i k_1 x + i k_2 x} \\ \times \delta^3(k_1 - q_1) \delta^3(k_2 - q_2)$$

$$\Rightarrow M = -ig \int d^4 x e^{-i(p - q_1 - q_2)x}$$

Tony
(3.30)
p56

$$M = -ig \delta^4(p - q_1 - q_2)$$

- A first scattering amplitude $\Rightarrow$ Probability of SC $\propto g^2$
- Its zero or ∞

(But this is not directly measurable.)

Have to integrate over range of momenta, flavors etc - a probability density.

co L16
16/11/15

Physical States & Vacuum Expectation Values

Consider our initial state

$$\begin{aligned} |i\rangle &= |\phi(p)\rangle \\ &= \sqrt{2\omega_p} \hat{a}_p^\dagger |0\rangle \end{aligned}$$

We may rewrite this in terms of $\hat{\phi}|0\rangle$

First note

$$\begin{aligned} \hat{\phi}(y)|0\rangle &= \int d^3k \frac{(\hat{a}_k e^{-iky} + \hat{a}_k^\dagger e^{iky})}{\sqrt{2\omega_k}} |0\rangle \\ &= \int d^3k \frac{e^{iky}}{\sqrt{2\omega_k}} |k\rangle \end{aligned}$$

To project out just $k=p$ mode use

$$\begin{aligned} & \left(\int d^3y e^{-ipy} 2\omega_p \right) \times (\hat{\phi}(y)|0\rangle) \\ &= \int d^3k \frac{2\omega_p}{\sqrt{2\omega_k}} \underbrace{\int d^3y e^{-i(p-k)y}}_{\delta^3(p-k)} |k\rangle e^{-i(\omega_p - \omega_k)t} \end{aligned}$$

$$= \sqrt{2\omega_p} |p\rangle = |p\rangle$$

Relativistically
Normalised
State

This allows us to link

MATRIX ELEMENTS M

$$M = \langle F | S | i \rangle$$

to

to

$$\underbrace{a^\dagger, b^\dagger, c^\dagger | 0 \rangle}$$

GREEN FUNCTIONS G

vacuum expectation values
of fields

$$G = \langle 0 | T \{ \hat{\phi}(x_1) \hat{\phi}^\dagger(x_2) \hat{\phi}(x_3) \dots | 0 \rangle$$

Rule

For every particle of momentum p
in initial state ($p_0 = \omega_p > 0$)

replace $|p\rangle = \sqrt{2\omega_p} \hat{a}^\dagger |0\rangle$ by

$$\text{SKIP} \left[\left(\int d^3y e^{-ip \cdot y} 2\omega_p \right) \underbrace{\hat{\phi}^\dagger(y)}_{\text{Choose } y_0 \rightarrow -\infty} |0\rangle \right]_{\text{SKIP}}$$

$$\Rightarrow M = \prod_F \left(\int d^3z_f e^{+ip_f \cdot z_f} 2\omega_{p_f} \right) \prod_i \left(\int d^3y_i e^{-ip_i \cdot y_i} 2\omega_{p_i} \right) G(\{z\}, \{y\})$$

$$G(\{z\}, \{y\}) = \langle 0 | T (\text{final}) S (\text{initial}) | 0 \rangle$$

e.g. for our $\phi \rightarrow \psi\psi$ decay we need

$$\mathcal{M} = \langle \bar{\psi}(q_2), \psi(q_1) | S | \phi(p) \rangle$$

$$= \pi \left(\int d^3 z_f e^{+i q_f z_f} 2\Omega_{p_f} \right)$$

$$\left(\int d^3 y e^{-i p y} 2\omega_p \right)$$

$$G(z_1, z_2, y)$$

$$\left(\begin{matrix} z_1^0, z_2^0 \rightarrow +\infty \\ y^0 \rightarrow -\infty \end{matrix} \right)$$

where

$$G(z_1, z_2, y)$$

$$= \langle 0 | T \hat{\psi}(z_1) \hat{\psi}^\dagger(z_2) \hat{S} \hat{\phi}(y) | 0 \rangle$$

For perturbation theory we expand S in powers of coupling constants so

$$G = \sum \underbrace{\langle 0 | T (\text{fields}) | 0 \rangle}$$

Each term represented as part of one Feynman diagram

Let's see how...

we can